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# On Ternary Monoid of Hypersubstitutions of Type $\tau = (n)$

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## ABSTRACT

A ternary monoid of hypersubstitutions of type $\tau = (n)$ is the set $Hyp(n)$ of all hypersubstitutions of type $\tau = (n)$ with a ternary operation which satisfies the associative law, and has the identity element $\sigma_{id}$ . For $n = 2$ , the idempotent and regular elements, the ideals of submonoids and some algebraic-properties of this monoid were studied by author. In this present paper, we study the algebraic-structural properties of $Hyp(n), n > 2$ and characterize the idempotent and regular elements. In particular, we describe the relationships between some submonoids of this monoid under the ideal of this submonoids.

**Keywords:** hypersubstitutions, ternary monoid, ternary ideal.

# 1. Introduction

There are many research papers about pure mathematics that benefits to the further studies. Algebra is the one of pure mathematics which is a basic concept of mathematics that can be extent to advance mathematics. The algebraic-structural properties of algebra was studied by many authors.

A hypersubstitution is one of algebra that was studies by many authors. Denecke and Koppitz (1998) studied on the monoid of hypersubstitutions of type $\tau = (2)$ by studying the finite monoid of hypersubstitutions of type $\tau = (2)$. They determined all finite submonoids and studied some properties of this monoid. Wishmath (2000) extended the concept of Denecke and Koppitz (1998) by studied the monoid of hypersubstitutions of type $\tau = (n), n > 2$. They described some algebraic-structure properties and characterized dual, projection and idempotent of this monoid.

The topic on algebra that interested by many authors is ternary semigroup which was firstly investigated by Kasner (1904). Ternary semigroups are special case of $n$-ary semigroups when we put $n = 3$. Later, Siosan (1963) gave the description of the regular algebraic system with respect to the $m$-ary operation. After that, Siosan (1965b) studied the ideals in $(m + 1)$-semigroups and the ideal theory in ternary semigroups was introduced by Siosan (1965a). A new definition of regular $n$-semigroups was described by Dudek and Groúdzińska (1980). They proved some theorem of a regular $n$-semigroups and studied the ideal theory on it. Later, Dudek (2001) described the idempotent in $n$-ary semigroups. The notion of congruences on ternary semigroup was introduced by Kar and Maity (2007). They studied some of its interested properties and characterized the cancellative congruence, group congruence, and Rees congruence on ternary semigroup. Santiago and Bala (2010) studied the regularity conditions in ternary semigroup by comparing with a semigroup. They obtained that many properties in semigroup are hold in ternary semigroup. Iampan (2013) introduced the basic concept of an ideal in a ternary semigroup and studied the interesting properties of the ideal of ternary semigroup. Moreover, they described the relationships between ideal and semilattices congruence on ternary semigroup.

In this present paper, we use the concept above to be constructing the ternary monoid of hypersubstitution of type $\tau = (n)$ and study algebraic-structural properties and special elements of this monoid. After that, we describe the relationships between some submonoids of this monoid under the ideal of this submonoids.

# 2. Definitions and Notations

## 2.1 Ternary Semigroup

**Definition 2.1.** *Let $T$ be a nonempty set. Defined a ternary operation $[-,-,-] : T^3 \to T$ on $T$ by $(x_1, x_2, x_3) \longmapsto [x_1x_2x_3]$. Then $(T, [-,-,-])$ is called* ***a ternary semigroup*** *if it satisfies the following associative law:*

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

*for all $x_1, x_2, x_3, x_4, x_5 \in T$.*

Let $(T, \cdot)$ be a semigroup. A ternary operation on $T$ is defined by $[x_1x_2x_3] = x_1 \cdot x_2 \cdot x_3$ for any $x_1, x_2, x_3 \in T$. Then $T$ is a ternary semigroup.

**Definition 2.2.** *Let $T$ be a ternary semigroup. A nonempty subset $A$ of $T$ is said to be* ***a ternary subsemigroup*** *if and only if $AAA \subseteq A$.*

**Definition 2.3.** *Let $T$ be a ternary semigroup. An element $a \in T$ is said to be an* ***identity*** *if $[aas] = [asa] = [saa]$, for all $s \in T$.*

**Definition 2.4.** *Let $T$ be a ternary semigroup and $a \in T$.*

*(i) An element $a$ is said to be and* ***idempotent*** *provided that $[aaa] = a^3 = a$, for any $a \in T$.*

*(ii) An idempotent element $a$ which is not the identity element is called a* ***proper idempotent****.*

**Definition 2.5.** *Let $T$ be a ternary semigroup. An element $a \in T$ is said to be* ***regular*** *if there exist $x, y \in T$ such that $[[axa]ya] = axaya = a$.*

**Definition 2.6.** *A nonempty subset $A$ of a ternary semigroup $T$ is said to be*

*(i) a* ***left ideal*** *of $T$ if $b, c \in T, a \in A$ implies $bca \in A$, i.e. $TTA \subseteq A$.*

*(ii) a* ***lateral ideal*** *of $T$ if $b, c \in T, a \in A$ implies $bac \in A$, i.e. $TAT \subseteq A$.*

*(iii) a* ***right ideal*** *of $T$ if $b, c \in T, a \in A$ implies $abc \in A$, i.e. $ATT \subseteq A$.*

*(iv) an* ***ideal*** *of $T$ if $A$ is a left-, lateral-, and right-ideal of $T$.*

## 2.2 Hypersubstitutions

**Definition 2.7.** *Let $f_i$ be $n_i$-ary operation symbols which $n_i \in \mathbf{N}$ and $X_n = \{x_1, \cdots, x_n\}$ be a set of variables. The $n$-**ary term** of type $\tau$ are inductively defined by the following way:*

*(i) Every variable $x_i \in X_n$ is an $n$-ary term.*

*(ii) If $t_1, \cdots, t_{n_i}$ are $n$-ary terms, then $f_i(t_1, \cdots, t_{n_i})$ is an $n$-ary term.*

Denoted by $W_\tau(X)$ the set of all $n$-ary terms of type $\tau$.

**Definition 2.8.** *Let $\{f_i \mid i \in I\}$ be the set of all $n_i$-ary operation symbols and $W_\tau(X_n)$ be the set of all $n$-ary terms of type $\tau$. A **hypersubstitution of type** $\tau$ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X_n)$ which preserves arities.*

We denoted by $Hyp(\tau)$ the set of all hypersubstitutions of type $\tau$.

**Definition 2.9** (Superposition Operation)**.** *Let $W_\tau(X_n), W_\tau(X_m), m, n \in \mathbf{N}$ be the set of all $n$-ary and $m$-ary terms of type $\tau = (n_i)_{i \in I}$. The superposition of terms $S_m^n : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$ is inductively defined as follows:*

*(i) $S_m^n(x_i, t_1, \ldots, t_n) := t_i$ where $x_i \in X_n, t_1, \ldots, t_n \in W_\tau(X_m)$*

*(ii) $S_m^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S_m^n(s_1, t_1, \ldots, t_n), \ldots, S_m^n(s_{n_i}, t_1, \ldots, t_n))$ where $f_i(s_1, \ldots, s_{n_i}) \in W_\tau(X_n)$.*

For any $\sigma \in Hyp(\tau)$, the extension $\hat{\sigma}$ of $\sigma$ is a mapping $\hat{\sigma} : W_\tau(X_n) \to W_\tau(X_n)$ where $\hat{\sigma}, t \in W_\tau(X)$ define inductively by

(i) $\hat{\sigma}[x] := x$, for any variable $x \in X$ and

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S_m^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ where $\hat{\sigma}[f_j]; 1 \leq j \leq n_i$ are already defined.

Then the binary operation $\circ_h$ on $Hyp(\tau)$ is defined by $\sigma_1 \circ_h \sigma_2 = \hat{\sigma_1} \circ \sigma_2$ for any$\sigma_1, \sigma_2 \in Hyp(\tau)$ and $\circ$ is the usual composition of mappings.

**Proposition 2.1.** *(Koppitz and Denecke (2006)) If $\hat{\sigma}$ is the extension of a hypersubstitution $\sigma$, then for $n, m \in \mathbf{N}$*

$$\hat{\sigma}[S_n^n(t, t_1, \ldots, t_n)] = S_n^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).$$

**Proposition 2.2.** *(Koppitz and Denecke (2006)) Let $\sigma_1, \sigma_2 \in Hyp(\tau)$. Then $\hat{\sigma_1} \circ \sigma_2$ is a hypersubstitution, and*

$$(\hat{\sigma_1} \circ \sigma_2)\hat{} = \hat{\sigma_1} \circ \hat{\sigma_2}.$$

Throughout this present paper, we denote:

$\sigma_t :=$ the hypersubstitution $\sigma$ of type $\tau$ which maps $f$ to the term $t$,

$var(t)$:= the set of all variables occurring in the term $t$,

$op(t) :=$ the number of all operation symbols occuring in the term $t$,

$firstop(t) :=$ the first operation symbols occuring in the term $t$ is $f$,

$W_n(X_n) = \{t \mid x_1, \cdots, x_n \in var(t)\}$.

## 3. Ternary Monoid of Hypersubstitutions of type $\tau = (n)$

In this section, we first give the concept of the ternary monoid $Hyp(n)$ to study algebraic-structural properties of this monoid as following.

Let $[-,-,-] : (Hyp(n))^3 \to Hyp(n)$ be a ternary operation defined by $[\sigma_1\sigma_2\sigma_3] = \sigma_1 \circ_h \sigma_2 \circ_h \sigma_3$ for any $\sigma_1, \sigma_2, \sigma_3 \in Hyp(n)$. Then we have $\underline{Hyp(n)} := (Hyp(n), [-,-,-])$ is a ternary semigroup.

Let $\sigma_t \in Hyp(n)$ where $t \in W_{(n)}(X_n)$, and $\sigma_{id} = \sigma_{f(x_1,\cdots,x_n)}$. Then we have

$$\begin{aligned}
[\sigma_{id}\sigma_{id}\sigma_t](f) &= (\sigma_{f(x_1,\cdots,x_n)} \circ_h \sigma_{f(x_1,\cdots,x_n)} \circ_h \sigma_t)(f) \\
&= \hat{\sigma}_{f(x_1,\cdots,x_n)}[\hat{\sigma}_{f(x_1,\cdots,x_n)}[\sigma_t(f)]] \\
&= \hat{\sigma}_{f(x_1,\cdots,x_n)}[t] \\
&= \sigma_t(f).
\end{aligned}$$

Similarly. we obtain that $[\sigma_{id}\sigma_t\sigma_{id}](f) = \sigma_t(f) = [\sigma_t\sigma_{id}\sigma_{id}](f)$.

**Proposition 3.1.** $\underline{Hyp(n)} := (Hyp(n), [-,-,-], \sigma_{id})$ *is a ternary monoid.*

## 3.1 Projection and Dual hypersubstitutions of type $\tau = (n)$

We first consider the set of special elements of ternary monoid $Hyp(n)$. Let $\sigma_t \in Hyp(n)$, we called $\sigma_t$ a **projection hypersubstitution** if $t$ is a variable. The set of all projection hypersubstitutions is denoted by $P(n) = \{\sigma_{x_i} : x_i \in X_n\}$.

**Lemma 3.1.** *Let $\sigma_t$ be in $Hyp(n)$ and any $\sigma_{x_i}, \sigma_{x_j} \in P(n), 1 \leq i, j \leq n$ which $i \neq j$. Then the following holds.*

*(i)* $[\sigma_t \sigma_{x_i} \sigma_{x_i}] = [\sigma_{x_i} \sigma_t \sigma_{x_i}] = [\sigma_t \sigma_t \sigma_{x_i}] = \sigma_{x_i}$.

*(ii)* $[\sigma_t \sigma_{x_i} \sigma_{x_j}] = [\sigma_{x_i} \sigma_t \sigma_{x_j}] = \sigma_{x_j}$.

*(iii)* $[\sigma_{x_i} \sigma_{x_j} \sigma_t] \in P(n)$.

*(iv)* $[\sigma_{x_i} \sigma_t \sigma_t] \in P(n)$.

*Proof.* (i) Let $\sigma_t \in Hyp(n)$ and $\sigma_{x_i} \in P(n)$. We have

$$[\sigma_t \sigma_{x_i} \sigma_{x_i}](f) = (\sigma_t \circ_h \sigma_{x_i} \circ_h \sigma_{x_i})(f) = \hat{\sigma}_t[\hat{\sigma}_{x_i}[\sigma_{x_i}(f)]] = \hat{\sigma}_t[x_i] = \sigma_{x_i}(f).$$

Similarly, we have $[\sigma_{x_i} \sigma_t \sigma_{x_i}](f) = \sigma_{x_i}(f) = [\sigma_t \sigma_t \sigma_{x_i}](f)$.

(ii) The proof is similarly to (i).

(iii) We will proceed by induction on the coplexity of term $t$. If $t \in X_n$, then, by (i) and (ii), we obtain that $[\sigma_{x_i} \sigma_{x_j} \sigma_t](f) = \sigma_t(f) \in P(n)$.

Assume that $t = f(u_1, \cdots, u_n)$, and $\sigma_{x_i} \circ_h \sigma_{u_1}, \cdots, \sigma_{x_i} \circ_h \sigma_{u_n} \in P(n)$. Consider

$$\begin{aligned} [\sigma_{x_i} \sigma_{x_j} \sigma_t](f) &= (\sigma_{x_i} \circ_h \sigma_{x_j} \circ_h \sigma_{f(u_1, \cdots, u_n)})(f) \\ &= \hat{\sigma}_{x_i}[S^n_m(x_j, \hat{\sigma}_{x_j}[u_1], \cdots, \hat{\sigma}_{x_j}[u_n])] \\ &= \hat{\sigma}_{x_i}[\hat{\sigma}_{x_j}[u_j]]. \end{aligned}$$

Since $i = j$, then $[\sigma_{x_i} \sigma_{x_j} \sigma_t] = \hat{\sigma}_{x_i}[u_i] \in P(n)$. Since $i \neq j$, then $[\sigma_{x_i} \sigma_{x_j} \sigma_t] = \hat{\sigma}_{x_i}[x_j] \in P(n)$. Therefore, $[\sigma_{x_i} \sigma_{x_j} \sigma_t] \in P(n)$.

(iv) The proof is similarly to (iii).

□

**Lemma 3.2.** *For any $\sigma_{t_1}, \sigma_{t_2} \in Hyp(n)$ where $t_1 \neq t_2$ and $\sigma_{x_i} \in P(n), 1 \leq i \leq n$ which $i \neq j$, we have* *(i)* $[\sigma_{t_1}\sigma_{t_2}\sigma_{x_i}] = [\sigma_{t_2}\sigma_{t_1}\sigma_{x_i}] = \sigma_{x_i}$,

*(ii)* $[\sigma_{t_1}\sigma_{x_i}\sigma_{t_2}], [\sigma_{t_2}\sigma_{x_i}\sigma_{t_1}] \in P(n)$,

*(iii)* $[\sigma_{x_i}\sigma_{t_1}\sigma_{t_2}], [\sigma_{x_i}\sigma_{t_2}\sigma_{t_1}] \in P(n)$.

*Proof.* (i) It is obviously from Lemma 3.1 (i).

(ii) If $t_2 \in X_n$, then, by Lemma 3.1 (i),(ii), we obtain that $[\sigma_{t_1}\sigma_{x_i}\sigma_{t_2}](f) = \sigma_{t_2}(f) \in P(n)$.

Assume that $t_2 = f(u_1, \cdots, u_n)$, and $\sigma_{x_i} \circ_h \sigma_{u_1}, \cdots, \sigma_{x_i} \circ_h \sigma_{u_n} \in P(n)$. If $t_1 \in X_n$, then, by Lemma 3.1 (iii), it is easy to see that $\sigma_{t_1}\sigma_{x_i}\sigma_{t_2} \in P(n)$. Assume that $t_1 = f(v_1, \cdots, v_n)$, we consider

$$\begin{aligned}
[\sigma_{t_1}\sigma_{x_i}\sigma_{t_2}](f) &= (\sigma_{f(v_1,\cdots,v_n)} \circ_h \sigma_{x_i} \circ_h \sigma_{f(u_1,\cdots,u_n)})(f) \\
&= \hat{\sigma}_{f(v_1,\cdots,v_n)}[S^n_m(x_i, \hat{\sigma}_{x_i}[u_1], \cdots, \hat{\sigma}_{x_i}[u_n])] \\
&= \hat{\sigma}_{f(v_1,\cdots,v_n)}[\hat{\sigma}_{x_i}[u_i]].
\end{aligned}$$

Since $\hat{\sigma}_{x_i}[u_i] \in P(n)$, then $[\sigma_{t_1}\sigma_{x_i}\sigma_{t_2}] \in P(n)$. By the same way, we can show that $[\sigma_{t_2}\sigma_{x_i}\sigma_{t_1}] \in P(n)$ .

(iii) The proof is similarly to (ii).

□

**Corollary 3.1.** *(i) $P(n) \cup \{\sigma_{id}\}$ is a ternary submonoid of $Hyp(n)$.*

*(ii) $P(n)$ is a ternary ideal of $Hyp(n)$.*

We next study a special kind of hypersubstitution in $Hyp(n)$ that are **dual hypersubstitutions**, which are define as following. Let $\pi$ be a permutation of the set $J = \{1, 2, \cdots, n\}$. For any such permutation $\pi$, we let $\sigma_\pi$ be the hypersubstitution such as $\sigma_\pi = \sigma_{f(x_{\pi(1)},\cdots,x_{\pi(n)})}$. The set of all such dual hypersunstitutions $\sigma_\pi$ denoted by $D(n)$.

**Lemma 3.3.** *For any permutations $\pi, \rho$ and $\gamma$, we have $[\sigma_\pi\sigma_\rho\sigma_\gamma] = \sigma_{\pi\circ\rho\circ\gamma}$.*

*Proof.* We consider

$$\begin{aligned}
[\sigma_\pi\sigma_\rho\sigma_\gamma](f) &= (\sigma_\pi \circ_h \sigma_\rho \circ_h \sigma_\gamma)(f) \\
&= \hat{\sigma}_\pi[\hat{\sigma}_\rho[f(x_{\gamma(1)}, \cdots, x_{\gamma(n)})]] \\
&= \hat{\sigma}_\pi[S^n_m(f(x_{\rho(1)}, \cdots, x_{\rho(n)}), x_{\gamma(1)}, \cdots, x_{\gamma(n)})] \\
&= \hat{\sigma}_\pi[f(x_{\rho(\gamma(1))}, \cdots, x_{\rho(\gamma(n))})] \\
&= S^n_m(f(x_{\pi(1)}, \cdots, x_{\pi(n)}), x_{\rho(\gamma(1))}, \cdots, x_{\rho(\gamma(n))}) \\
&= f(x_{\pi(\rho(\gamma(1)))}, \cdots, x_{\pi(\rho(\gamma(n)))}) = \sigma_{\pi\circ\rho\circ\gamma}(f).
\end{aligned}$$

□

**Lemma 3.4.** *Let $\sigma, \delta$ and $\eta$ be in $Hyp(n)$. If $[\sigma\delta\eta] \in D(n)$, then $\sigma, \delta$ and $\eta$ are in $D(n)$.*

*Proof.* Let $\sigma(f) = f(u_1, \cdots, u_n), \delta(f) = f(v_1, \cdots, v_n)$ and $\eta(f) = f(s_1, \cdots, s_n)$. Consider

$$\begin{aligned}
[\sigma\delta\eta](f) &= (\sigma \circ_h \delta \circ_h \eta)(f) \\
&= \hat{\sigma}[\hat{\delta}[f(s_1, \cdots, s_n)]] \\
&= \hat{\sigma}[S^n_m(f(v_1, \cdots, v_n), \hat{\delta}[s_1], \cdots, \hat{\delta}[s_n])] \\
&= \hat{\sigma}[f(S^n_m(v_1, \hat{\delta}[s_1], \cdots, \hat{\delta}[s_n]), \cdots, S^n_m(v_n, \hat{\delta}[s_1], \cdots, \hat{\delta}[s_n]))] \\
&= S^n_m(f(u_1, \cdots, u_n), \hat{\sigma}[S^n_m(v_1, \hat{\delta}[s_1], \cdots, \hat{\delta}[s_n])], \cdots, \\
&\quad \hat{\sigma}[S^n_m(v_n, \hat{\delta}[s_1], \cdots, \hat{\delta}[s_n])]) \\
&= f(S^n_m(u_1, \hat{\sigma}[S^n_m(v_1, \hat{\delta}[s_1], \cdots, \hat{\delta}[s_n])], \cdots, \hat{\sigma}[S^n_m(v_n, \hat{\delta}[s_1])], \\
&\quad \cdots, \hat{\delta}[s_n])], \cdots, S^n_m(u_n, \hat{\sigma}[S^n_m(v_1, \hat{\delta}[s_1], \cdots, \hat{\delta}[s_n])], \cdots, \\
&\quad \hat{\sigma}[S^n_m(v_n, \hat{\delta}[s_1])], \cdots, \hat{\delta}[s_n])]).
\end{aligned}$$

Since $[\sigma\delta\eta] \in D(n)$, then there exist a pemutation $\pi$ such that $[\sigma\delta\eta](f) = f(x_{\pi(1)}, \cdots, x_{\pi(n)})$.Thus

$$S^n_m(u_i, \hat{\sigma}[S^n_m(v_1, \hat{\delta}[s_1], \cdots, \hat{\delta}[s_n])], \cdots, \hat{\sigma}[S^n_m(v_n, \hat{\delta}[s_1])]) = x_{\pi(i)}.$$

Since $\pi$ is a permutation, this force all $u_i$'s, $v_i$'s and $s_i$'s to be distinct variables. Therefore, $\sigma, \delta$ and $\eta$ are in $D(n)$. □

## 3.2 Idempotent and Regular elements of ternary monoid $Hyp(n)$

All idempotent elements of the monoid $Hyp(n)$ was studied by Wishmath (2000). In this section, we characterize the idempotent and regular element in the ternary monoid $Hyp(n)$. For study the idempotent elements, we first give the following notations.

$F$:= the set of hypersubstitutions of the form $\sigma_{f(x_i,\cdots,x_i)}, 1 \leq i \leq n$,

$W_{x_i}$:= the set of terms of type $\tau = (n)$ constructed only from the variable $x_i$,

$E_{x_i} := \{\sigma_t : t = f(u_1, \cdots, u_n)$ where $u_i = x_i$ and all $u_j \in W_{x_i}\}$,

$E = E_{x_1} \cup \cdots \cup E_{x_n}$,

$E^*_{x_i} := \{\sigma_t : t = f(u_1, \cdots, u_n)$ where $u_i \in W_{x_i} \setminus \{x_i\}\}$,

$E^* = E^*_{x_1} \cup \cdots \cup E^*_{x_n}$.

**Lemma 3.5.** *For any $\sigma_s, \sigma_t$ and $\sigma_r$ of $E_{x_i}, 1 \leq i \leq n$, we have $[\sigma_s\sigma_t\sigma_r] = \sigma_s$.*

*Proof.* Let $s = f(u_1, \cdots, u_n), t = f(v_1, \cdots, v_n)$ and $r = f(w_1, \cdots, w_n)$ where $u_i = v_i = w_i = x_i$ and all $u_j, v_j$ and $w_j$ using only the variable $x_i$. Consider

$$\begin{aligned}
[\sigma_s\sigma_t\sigma_r](f) &= (\sigma_s \circ_h \sigma_t \circ_h \sigma_r)(f) \\
&= \hat{\sigma}_s[\hat{\sigma}_t[f(w_1, \cdots, w_n)]] \\
&= \hat{\sigma}_s[S^n_m(f(v_1, \cdots, v_n), \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n])] \\
&= \hat{\sigma}_s[f(S^n_m(v_1, \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n]), \cdots, S^n_m(v_n, \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n])] \\
&= S^n_m(f(u_1, \cdots, u_n), \hat{\sigma}_s[S^n_m(v_1, \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n])], \cdots, \\
&\quad \hat{\sigma}_s[S^n_m(v_n, \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n])]) \\
&= f(S^n_m(u_1, \hat{\sigma}_s[S^n_m(v_1, \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n])], \cdots, \hat{\sigma}_s[S^n_m(v_n, \hat{\sigma}_t[w_1], \\
&\quad \cdots, \hat{\sigma}_t[w_n])], \cdots, S^n_m(u_n, \hat{\sigma}_s[S^n_m(v_1, \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n])], \cdots, \\
&\quad \hat{\sigma}_s[S^n_m(v_n, \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n])])).
\end{aligned}$$

Since $u_i = v_i = w_i = x_i$, the $i$th entry in this term is $x_i$. For any $j, 1 \leq j \leq n$, the $j$th entry in this term use only $x_i$, which is $\hat{\sigma}_t[w_i] = x_i$. This implies that $\hat{\sigma}_s[v_i] = x_i$. Thus the $j$th entry is $S^n_m(v_j, \hat{\sigma}_t[w_1], \cdots, \hat{\sigma}_t[w_n]) = S^n_m(v_j, x_i, \cdots, x_i) = v_j$. So, $S^n_m(u_j, \hat{\sigma}_s[v_1], \cdots, \hat{\sigma}_s[v_n]) = S^n_m(u_j, x_i, \cdots, x_i) = u_j$. Therefore, $[\sigma_s\sigma_t\sigma_r](f) = f(u_1, \cdots, u_n) = \hat{\sigma}_s(f)$. $\square$

It is easy to see that all idempotent elements in the monoid $Hyp(n)$ are ternary idempotent, but the converse is not hold. For trivial example, $\sigma_{f(x_n,\cdots,x_1)}$ is a ternary idempotent, but not an idempotent in $Hyp(n)$. Trivially, we also see that every idempotent element is regular in the monoid $Hyp(n)$.

**Corollary 3.2.** *A hypersubstitution $\sigma$ is a proper ternary idempotent element of a ternary monoid $Hyp(n)$ iff $\sigma \in P(n) \cup F \cup E \cup E^* \cup \{\sigma_{f(x_n,\cdots,x_1)}\}$.*

**Lemma 3.6.** *Let $\sigma_t \in Hyp(n)$ where $t = f(u_1, \cdots, u_n)$. Then the following ststements hold.*

*(i) If $u_i \in X_n$ and all $u_j \in W_n(X_n) \setminus X_n$, then $\sigma_t$ is not ternary idempotent.*

*(ii) If $u_i \in W_n(X_n) \setminus X_n$ and all $u_j \in X_n$, then $\sigma_t$ is not ternary idempotent.*

*(iii) If $u_i \in W_n(X_n) \setminus X_n, 1 \leq i \leq n$, then $\sigma_t$ is not ternary idempotent.*

*Proof.* (i) Let $\sigma_t \in Hyp(n)$ where $t = f(u_1, \cdots, u_n)$. We consider

$$\begin{aligned} [\sigma_t\sigma_t\sigma_t](f) &= (\sigma_t \circ_h \sigma_t \circ_h \sigma_t)(f) \\ &= \hat{\sigma}_{f(u_1,\cdots,u_n)}[S^n_m(f(u_1, \cdots, u_n), \hat{\sigma}_{f(u_1,\cdots,u_n)}[u_1], \cdots, \\ &\quad \hat{\sigma}_{f(u_1,\cdots,u_n)}[u_n])]. \end{aligned}$$

Since $u_i \in X_n$, we have to substitute $u_i$ by $\hat{\sigma}_t[u_i]$, which is $x_i$. For any $u_j$, we have to substitute by $\hat{\sigma}_{f(u_1,\cdots,u_n)}[u_j], u_j \in W_n(x_n) \setminus X_n$. Thus $op(\hat{\sigma}_{f(u_1,\cdots,u_n)}[S^n_m(f(u_1, \cdots, u_n), \hat{\sigma}_{f(u_1,\cdots,u_n)}[u_1], \cdots, \hat{\sigma}_{f(u_1,\cdots,u_n)}[u_n])]) > op(\sigma_{f(u_1,\cdots,u_n)})$. Hence $[\sigma_{f(u_1,\cdots,u_n)}\sigma_{f(u_1,\cdots,u_n)}\sigma_{f(u_1,\cdots,u_n)}](f) \neq \sigma_{f(u_1,\cdots,u_n)}(f)$. Therefore, $\sigma_t$ is not a ternary idempotent.

The proof of (ii) and (iii) are similarly. □

Next, we study the ternary regular element in $Hyp(n)$. Trivially, every regular element is ternary regular element. Moreover, the set of all ternary idempotent is also ternary regular, and we have the following properties.

**Corollary 3.3.** *Let $\sigma_t \in Hyp(n)$ where $t = f(u_1, \cdots, u_n)$. Then the following ststements hold.*

*(i) If $u_i \in X_n$ and all $u_j \in W_n(X_n) \setminus X_n$, then $\sigma_t$ is not ternary regular.*

*(ii) If $u_i \in W_n(X_n) \setminus X_n$ and all $u_j \in X_n$, then $\sigma_t$ is not ternary regular.*

*(iii) If $u_i \in W_n(X_n) \setminus X_n, 1 \leq i \leq n$, then $\sigma_t$ is not ternary regular.*

### 3.3 Ternary ideal of submonoids of ternary monoid $Hyp(n)$

This section describes the relationships between submonoids of a ternary monoid $Hyp(n)$ under the ideal of this submonoid. We first recall the set of element in $Hyp(n)$ that use in this study as follow. Let

$P(n) := \{\sigma_t : t \text{ is a variable}\}$,

$D(n) := \{\sigma_\pi : \sigma_\pi(f) = f(x_{\pi(1)}, \cdots, x_{\pi(n)})\}$ where $\pi$ is a permutation,

$F$:= the set of hypersubstitutions of the form $\sigma_{f(x_i,\cdots,x_i)}, 1 \leq i \leq n$,

$M := F \cup D(n) \cup P(n)$,

$E_{x_i} := \{\sigma_t : t = f(u_1, \cdots, u_n) \text{ where } u_i = x_i \text{ and all } u_j \in W_{x_i}\}$,

$E = E_{x_1} \cup \cdots \cup E_{x_n}$,

$Reg(n) := \{\sigma : var(\sigma(f)) = \{x_1, \cdots, x_n\}\}$,

$Hypreg(n) := Reg(n) \cap \{\sigma : firstops(\sigma) = f\}$.

Then the following lemmas are the ideals of submonoids of $Hyp(n)$.

**Lemma 3.7.** *$P(n) \cup \{\sigma_{id}\}$, $F \cup \{\sigma_{id}\}$, $E_{x_i} \cup \{\sigma_{id}\}$, $D(n)$, $M$, $Reg(n)$ and $Hypreg(n)$ are ternary submonoids of a ternary monoid $Hyp(n)$.*

*Proof.* The proof of this lemma is straightforward. □

**Lemma 3.8.** *$F$ is a right ideal of $E_{x_i}$.*

*Proof.* Let $\sigma_t \in F$ and $\sigma_r, \sigma_s \in E_{x_i}$. Then $t = f(x_i, \cdots, x_i)$ and $s = f(s_1, \cdots, s_n)$, $r = f(r_1, \cdots, r_n)$ where $s_i = r_i = x_i$ and all $s_j, r_j \in W_{x_i}$. Consider

$$\begin{aligned} [\sigma_t\sigma_s\sigma_r](f) &= (\sigma_t \circ_h \sigma_s \circ_h \sigma_r)(f) \\ &= \hat{\sigma}_t[S^n_m(f(s_1, \cdots, s_n), \hat{\sigma}_s[r_1], \cdots, \hat{\sigma}_s[r_n])] \\ &= \hat{\sigma}_t[f(u_1, \cdots, u_n)] \text{ where } u_i \in W_{x_i} \\ &= S^n_m(f(x_i, \cdots, x_i), \hat{\sigma}_t[u_1], \cdots, \hat{\sigma}_t[u_n]). \end{aligned}$$

Since $s_i = r_i = x_i$, so we have $u_i = x_i$. This implies that $\hat{\sigma}_t[u_i] = x_i$. Thus $[\sigma_t\sigma_s\sigma_r](f) = S^n_m(f(x_i, \cdots, x_i), \hat{\sigma}_t[u_1], \cdots, \hat{\sigma}_t[u_n]) = f(x_i, \cdots .x_i) = \sigma_{f(x_i,\cdots,x_i)}(f) \in F$. Therefore, $F$ is a right ideal of $E_{x_i}$. □

**Lemma 3.9.** *$F$ is a right ideal of $F \cup E_{x_i}$.*

*Proof.* Let $\sigma_t \in F$ and $\sigma_r, \sigma_s \in F \cup E_{x_i}$. Then $t = f(x_i, \cdots, x_i)$. We will consider the following cases.

**Case 1.** If $\sigma_r, \sigma_s \in F$, then it is trivially because $F$ is a submonoid of $Hyp(n)$.

**Case 2.** If $\sigma_r, \sigma_s \in E_{x_i}$, then, by Lemma 3.8, we have $[\sigma_t \sigma_s \sigma_r](f) \in F$.

**Case 3.** If $\sigma_r \in F, \sigma_s \in E_{x_i}$, then $r = f(x_i, \cdots, x_i)$, and $s = f(s_1, \cdots, s_n)$ where $s_i = x_i$ and all $s_j \in W_{x_i}$. So

$$\begin{aligned} [\sigma_t \sigma_s \sigma_r](f) &= (\sigma_t \circ_h \sigma_s \circ_h \sigma_r)(f) \\ &= \hat{\sigma}_t[S^n_m(f(s_1, \cdots, s_n), \hat{\sigma}_s[x_i], \cdots, \hat{\sigma}_s[x_i])] \\ &= \hat{\sigma}_t[f(x_i, \cdots, x_i)] \\ &= f(x_i, \cdots, x_i) = \sigma_{f(x_i, \cdots, x_i)}(f) \in F. \end{aligned}$$

The case $\sigma_s \in F, \sigma_r \in E_{x_i}$ can proof similarly.

□

**Lemma 3.10.** *$F$ is an ideal of $D(n)$.*

*Proof.* Let $\sigma_t \in F$ and $\sigma_r, \sigma_s \in D(n)$. Then $\sigma_t = \sigma_{f(x_i, \cdots, x_i)}$ and $\sigma_s = \sigma_{f(x_{\pi(1)}, \cdots, x_{\pi(n)})}, \sigma_r = \sigma_{f(x_{\rho(1)}, \cdots, x_{\rho(n)})}$ where $\pi, \rho$ are permutatuons. Consider

$$\begin{aligned} [\sigma_s \sigma_r \sigma_t](f) &= (\sigma_s \circ_h \sigma_r \circ_h \sigma_t)(f) \\ &= \hat{\sigma}_s[S^n_m(f(x_{\rho(1)}, \cdots, x_{\rho(n)}), \hat{\sigma}_r[x_i], \cdots, \hat{\sigma}_r[x_i])] \\ &= \hat{\sigma}_s[f(x_i, \cdots, x_i)] \\ &= S^n_m(f(x_{\pi(1)}, \cdots, x_{\pi(n)}), \hat{\sigma}_s[x_i], \cdots, \hat{\sigma}_s[x_i]) \\ &= f(x_i, \cdots, x_i) = \sigma_{f(x_i, \cdots, x_i)}(f) \in F. \end{aligned}$$

And

$$\begin{aligned} [\sigma_s \sigma_t \sigma_r](f) &= (\sigma_s \circ_h \sigma_t \circ_h \sigma_r)(f) \\ &= \hat{\sigma}_s[S^n_m(f(x_i, \cdots, x_i), \hat{\sigma}_t[x_{\rho(1)}], \cdots, \hat{\sigma}_t[x_{\rho(n)}])] \\ &= \hat{\sigma}_s[f(x_{\rho(i)}, \cdots, x_{\rho(i)})] \\ &= S^n_m(f(x_{\pi(1)}, \cdots, x_{\pi(n)}), \hat{\sigma}_s[x_\rho(i)], \cdots, \hat{\sigma}_s[x_\rho(i)]) \\ &= f(x_{\pi(\rho(i))}, \cdots, x_{\pi(\rho(i))}) = \sigma_{f(x_{\pi(\rho(i))}, \cdots, x_{\pi(\rho(i))})}(f) \in F. \end{aligned}$$

Similarly, we have $[\sigma_t \sigma_s \sigma_r](f) \in F$. □

**Lemma 3.11.** *$F$ is an ideal of $M \setminus P(n)$.*

*Proof.* Let $\sigma_t \in F$ and $\sigma_r, \sigma_s \in M \setminus P(n)$. We will consider the following cases.

**Case 1.** If $\sigma_r, \sigma_s \in F$, then it is trivially because $F$ is a submonoid of $Hyp(n)$.

**Case 2.** If $\sigma_r, \sigma_s \in D(n)$, then, by Lemma 3.10, we have $[\sigma_s\sigma_r\sigma_t](f), [\sigma_s\sigma_t\sigma_r](f), [\sigma_t\sigma_s\sigma_r](f) \in F$.

**Case 3.** If $\sigma_r \in D(n), \sigma_s \in F$, then $r = f(x_{\pi(1)}, \cdots, x_{\pi(n)})$, and $s = f(x_i, \cdots, x_i)$ where $\pi$ is a permutation. So

$$\begin{aligned}
[\sigma_s\sigma_r\sigma_t](f) &= (\sigma_s \circ_h \sigma_r \circ_h \sigma_t)(f) \\
&= \hat{\sigma}_s[S^n_m(f(x_{\pi(1)}, \cdots, x_{\pi(n)}), \hat{\sigma}_r[x_i], \cdots, \hat{\sigma}_r[x_i])] \\
&= \hat{\sigma}_s[f(x_i, \cdots, x_i)] \\
&= f(x_i, \cdots, x_i) = \sigma_{f(x_i, \cdots, x_i)}(f) \in F.
\end{aligned}$$

And

$$\begin{aligned}
[\sigma_s\sigma_t\sigma_r](f) &= (\sigma_s \circ_h \sigma_t \circ_h \sigma_r)(f) \\
&= \hat{\sigma}_s[S^n_m(f(x_i, \cdots, x_i), \hat{\sigma}_t[x_{\pi(1)}], \cdots, \hat{\sigma}_t[x_{\pi(n)}])] \\
&= \hat{\sigma}_s[f(x_{\pi(i)}, \cdots, x_{\pi(i)})] \\
&= f(x_{\pi(i)}, \cdots, x_{\pi(i)}) = \sigma_{f(x_{\pi(i)}, \cdots, x_{\pi(i)})}(f) \in F.
\end{aligned}$$

Similarly, we have $[\sigma_s\sigma_r\sigma_t](f) \in F$. By the same way, we can proof the case $\sigma_r \in F, \sigma_s \in D(n)$.

□

**Lemma 3.12.** *$M \setminus D(n)$ is an ideal of $M$.*

*Proof.* The proof of this lemma is similarly to Lemma 3.11. □

**Lemma 3.13.** *$Hypreg(n)$ is an ideal of $Reg(n)$.*

*Proof.* Let $\sigma_t \in Hypreg(n)$ and $\sigma_s, \sigma_r \in Reg(n)$. Then $var(\sigma_t(f)) = var(\sigma_s(f)) = var(\sigma_r(f)) = \{x_1, \cdots, x_n\}$ and $firstops(\sigma_t(f)) = f$. We consider $[\sigma_s\sigma_r\sigma_t](f) = \hat{\sigma}_s[\hat{\sigma}_r[\sigma_t(f)]] = \hat{\sigma}_s[\hat{\sigma}_r[t]]$ where $t \in W_n(X_n) \setminus X_n$.

Since $\sigma_t \in Hypreg(n)$, then $var(t) = \{x_1, \cdots, x_n\}$ and $firstops(t) = f$. So, $var(\hat{\sigma}_s[\hat{\sigma}_r[t]]) = \{x_1, \cdots, x_n\}$ and $firstops(\hat{\sigma}_s[\hat{\sigma}_r[t]]) = f$. Thus $[\sigma_s\sigma_r\sigma_t](f) \in Hypreg(n)$.

Similarly, we obtain that $[\sigma_s\sigma_t\sigma_r](f), [\sigma_t\sigma_s\sigma_r](f) \in Hypreg(n)$. Therefore, $Hypreg(n)$ is an ideal of $Reg(n)$. □

# 4. Conclusion

In this paper, the ternary monoid of hypersubstitution is constructed. It is the set $\underline{Hyp(n)} := (Hyp(n), [-,-,-], \sigma_{id})$. There are regular and idempotent sets of elements of this ternary monoid. Some structural-properties of the special element of this monoid are presentd. The last of this paper, the relationships between some submoboid are described and presented as the ideal of submonoid.

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